Math 2E Quiz 4 Morning - April 21st

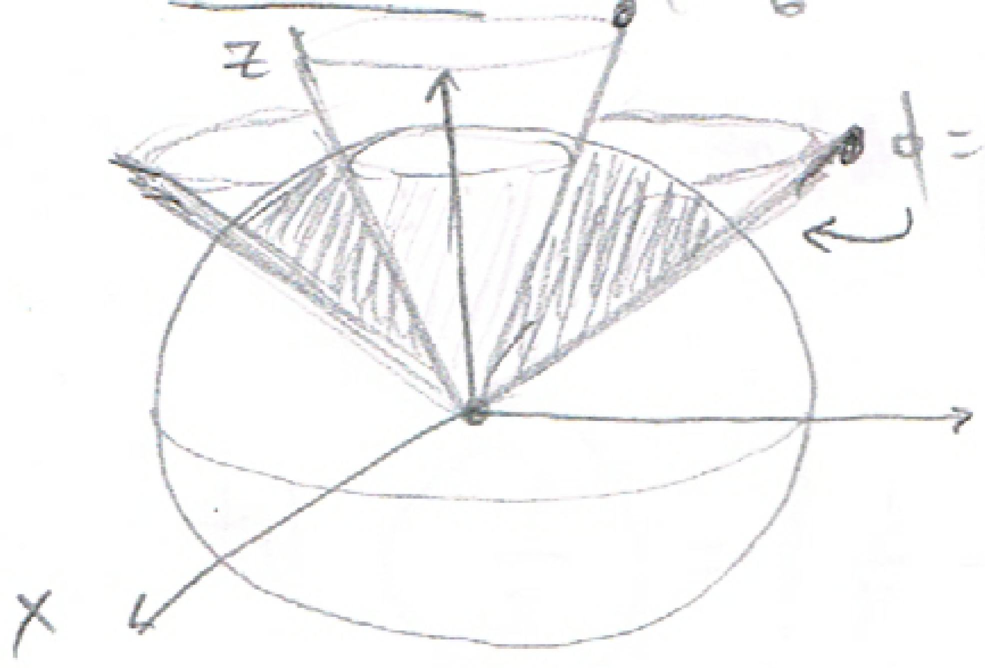
Please write your name and ID on the front.

Show all of your work, and simplify all your answers. *There is a question on the back side.

1. Find the volume of the part of the ball $x^2 + y^2 + z^2 \leq a^2$ that lies between the two cones $\phi = \pi/6$ and $\phi = \pi/3$.

(Not part of the problem, but maybe it will help with visualization, the cones in Cartesian are $\phi = \pi/3 \iff z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ and $\phi = \pi/6 \iff z = \sqrt{3}\sqrt{x^2+y^2}$.)

Picture: $\phi = \frac{\pi}{6}$, $z = \sqrt{3}\sqrt{x^2+y^2}$



$\phi = \frac{\pi}{3}$, $z = \sqrt{\frac{x^2+y^2}{3}}$

$\Rightarrow$ $\rho \in [0, a]$, $\theta \in [0, 2\pi]$, $\phi \in [\frac{\pi}{6}, \frac{\pi}{3}]$ +3 bounds

dV

* The volume technically goes all around from $0 \to 2\pi$ in θ.

So $V = \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\rho=0}^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ +2

Computing, $V = \int_{\theta=0}^{2\pi} \int_{\phi=\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\rho^3}{3}\Big|_0^a \sin\phi \, d\phi \, d\theta$

$= \frac{a^3}{3} \int_{\theta=0}^{2\pi} (-\cos\phi)\Big|_{\pi/6}^{\pi/3} d\theta = \frac{a^3}{3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) \int_0^{2\pi} d\theta$

$= \frac{\cancel{2}\pi a^3}{3}\left(\frac{-1+\sqrt{3}}{\cancel{2}}\right) = \boxed{\frac{\pi a^3}{3}(\sqrt{3}-1)}$ +5

2. Let R_{xy} be the region in $\mathbb{R}^2$ in the 1st quadrant bounded by the ellipse $9x^2+4y^2=1$. Consider

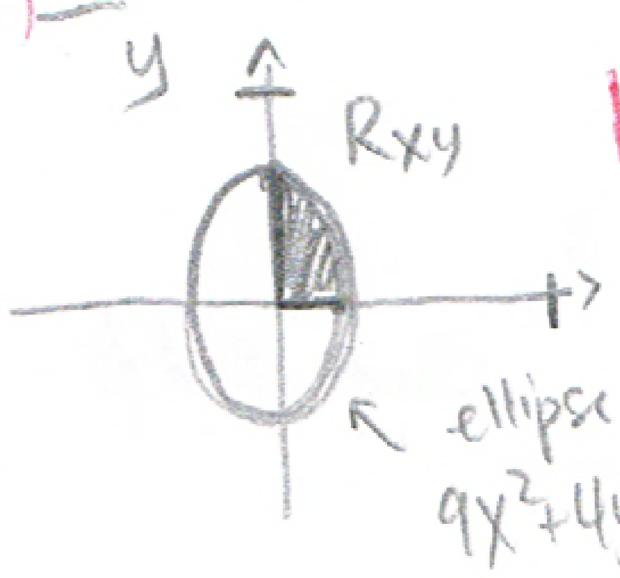
$$\iint\limits_{R_{xy}} \sin(9x^2+4y^2)\,dA.$$

With the change of variable $u=3x, v=2y$, we will rewrite this integral in uv-variables.
(i) What is the Jacobian with this transformation?
(ii) Draw the new region S_{uv} from applying the transformation to R_{xy}. Label the boundary curves.
(iii) Using (i) and (ii), write the integral in the uv variables, with correct bounds.
(iv) [*2pts Bonus*] Compute this integral.
You don't need to do part (iv) for full credit on the problem.

+2) (i) First, $x = u/3$, $y = v/2$ $\Rightarrow$ $J = \left|\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\right| = \left|\det\begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}\right| = \boxed{\frac{1}{6}}$

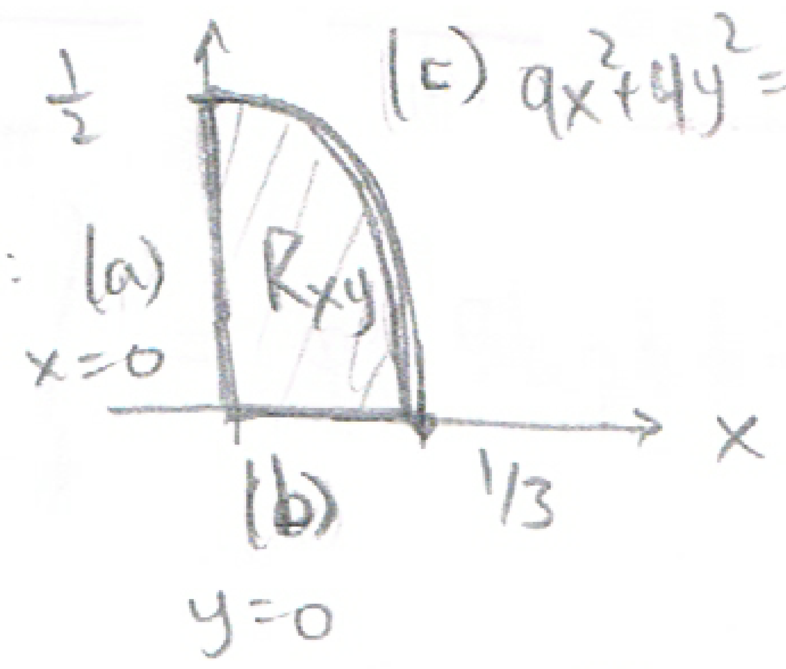
1 pt 1 pt

+4) (ii) We start with:



1 pt

, ZOOM in:

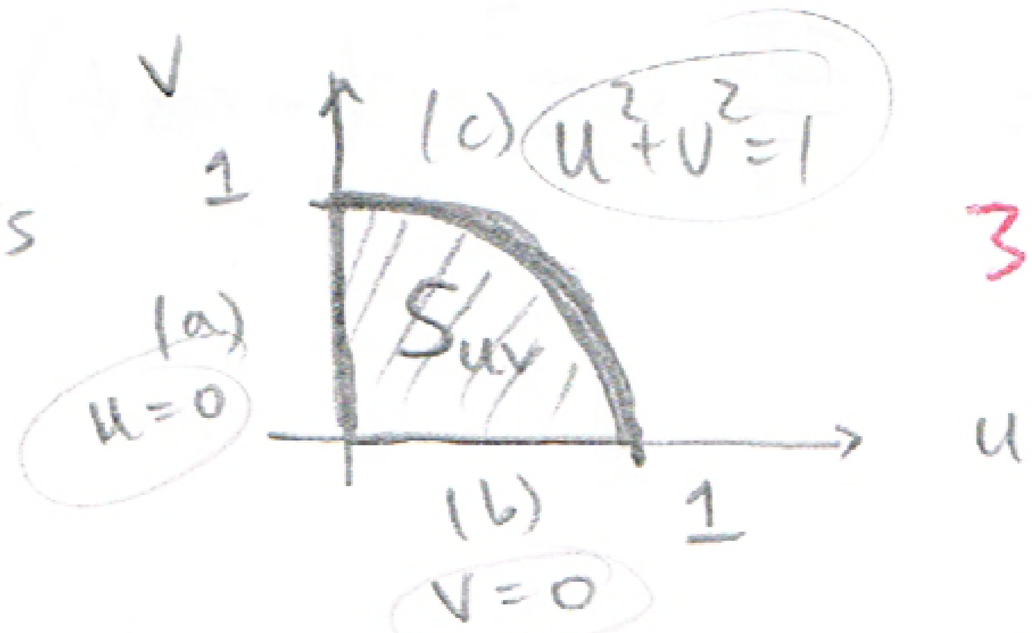


(a) $x=0$ from $0 \le y \le \frac{1}{2}$ implies $\boxed{u=0}$ from $0 \le v \le 1$

(b) $y=0$ from $0 \le x \le 1/3$ implies $\boxed{v=0}$ from $0 \le u \le 1$

(c) $9x^2+4y^2=1$ becomes $\boxed{u^2+v^2=1}$

So, our uv-domain is



3 pts

2 pts

+4) (iii) Using $x=\frac{u}{3}$, $y=\frac{v}{2}$, it's $\iint\limits_{S_{uv}} \sin(u^2+v^2)\cdot J\, dA_{uv} = \int_{u=0}^{1}\int_{v=0}^{\sqrt{1-u^2}} \sin(u^2+v^2)\,\frac{dv\,du}{6}$

1 pt 1 pt

Or in polar, $= \frac{1}{6}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} \sin(r^2)\, r\,dr\,d\theta.$

(iv) compute, it's θ-indep. $\underset{du=2rdr}{\overset{u=r^2}{=}} \frac{\pi}{2}\cdot\frac{1}{6}\int_{u=0}^{u=1} \sin(u)\cdot\frac{du}{2}$

(+2 EC)

$= \frac{\pi}{24}\left(-\cos(u)\right)\Big|_{u=0}^{1} = \boxed{\frac{\pi}{24}\left(1-\cos(1)\right)}$

(2pt EC)